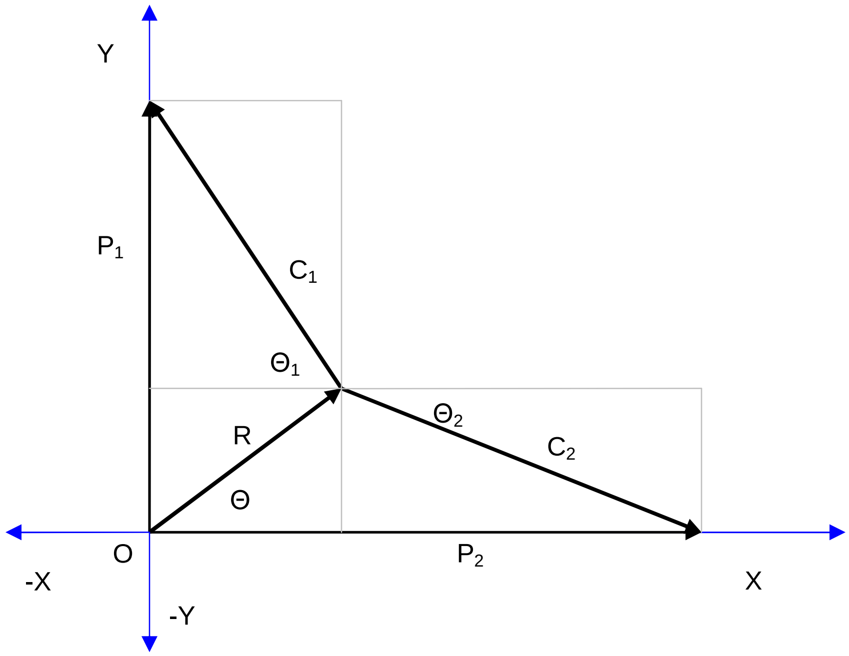
# Equations of Motion




Here we have the crankshaft at some angle **θ**, with two connecting rods attached to it. As defined before, the **R** vector points from the main journal centerline, to the rod journal centerline. The crankshaft rotates around O in a counterclockwise (X to Y) direction. The $\mathbf{C_1}$ and $\mathbf{C_2}$ vectors point from the rod journal centerline to the wrist pin centerline on the pistons. The $\mathbf{P_1}$ and $\mathbf{P_2}$ vectors point from the main journal centerline to the wrist pin centerline on the pistons.

The lengths of the **R**, $\mathbf{C_1}$ and $\mathbf{C_2}$ vectors are fixed by the geometry of the actual parts. We will define them later. The lengths of the $\mathbf{P_1}$ and $\mathbf{P_2}$ vectors vary and correspond to the position of the wrist pin in the stroke. Since the piston is attached, it is the piston position, as well.

The angles of the **R**, $\mathbf{C_1}$ and $\mathbf{C_2}$ vectors vary, according to the rotating position of the crankshaft. The angles of the $\mathbf{P_1}$ and $\mathbf{P_2}$ vectors are fixed, because the pistons are restrained to motion within the cylinders. The $\mathbf{C_1}$ and $\mathbf{C_2}$ vectors rotate back and forth from the wrist pins, to follow the end of the **R** vector. If you'll notice, the entire connecting rod, moves back and forth with the piston, so it is all part of the reciprocating mass. But we're not going to talk about mass yet.

The angles **θ**, $\boldsymbol{\theta_1}$, and $\boldsymbol{\theta_2}$ are all referenced from the same, horizontal direction, so that their relative values are accurately depicted in the model. The light grey lines are drawn in to help visualize the lengths of the sin (Y) and cos (X) components of each of the vectors. As stated earlier, the ability to look at X and Y components of the vectors will make the analysis much easier.

Next, notice that if you follow a path form the tail of the **R** vector, to the head of the $\mathbf{C_1}$ vector, you end up at the end of the $\mathbf{P_1}$ vector. Two paths to the same place. Same with the **R**, $\mathbf{C_2}$ and $\mathbf{P_2}$ vectors.

So let's call r the length of **R** and c the length of $\mathbf{C_1}$ and $\mathbf{C_2}$. Remember, a vector has magnitude and direction. I am just defining lengths, independent of direction.

The $\mathbf{P_1}$ and $\mathbf{P_2}$ vectors are conveniently aligned with the X and Y axes. The length of $\mathbf{P_1}$ is equal to the sum of the Y components of the **R** and $\mathbf{C_1}$ vectors. The X components of the **R** and $\mathbf{P_1}$vectors are equal in length, but opposite in direction, so their sum equals 0. This makes sense, since we know that the position of piston 1 is fixed in the X direction. The same logic can be used for **R**, $\mathbf{C_2}$ and $\mathbf{P_2}$.

With that, we can define the equations for the positions of the connecting rods and pistons for all rotational positions of the crankshaft by adding the X and Y components of the vectors:

$r\sin\theta + c\sin\theta_1 = p_1$
$r\cos\theta + c\cos\theta_1 = 0$
$r\sin\theta + c\cos\theta_2 = 0$
$r\cos\theta + c\cos\theta_2 = p_2$

So far, nothing moves. Next, we will take these position equations and find the velocities and accelerations that describe the motion. Then we'll be able to spin the crankshaft, speed it up, slow it down, and everything else will follow, according to these position rules.

You'll recall that velocity is the rate of change of position. This can be a linear position, or an angular position. Linear velocities will be designated with v values, angular velocities will be designated with ω (omega) values. You will also need to recall that angular velocity lags angular position by 90 degrees, as described previously. This means that if we are taking the sin component of position, the angular velocity will be a cos component, and the cos component will be a –sin component. Look at the X and Y axes to gain an understanding of positive and negative directions for sin and cos.

We also need to define the acceleration, which is the rate of change of velocity. Linear accelerations will be given a values, angular accelerations will be given α (alpha) values.

So here we go with the velocity and acceleration equations, based on rates of change of the position equations:

Position (repeated):

$r\sin\theta + c\sin\theta_1 = p_1$
$r\cos\theta + c\cos\theta_1 = 0$
$r\sin\theta + c\sin\theta_2 = 0$
$r\cos\theta + c\cos\theta_2 = p_2$

Velocity (rate of change of position):

$\omega r\cos\theta + \omega_1 c\cos\theta_1 = v_1$
$-\omega r\sin\theta - \omega_1 c\sin\theta_1 = 0$
$\omega r\cos\theta + \omega_2 c\cos\theta_2 =0$
$-\omega r\sin\theta - \omega_2 c\sin\theta_2 = v_2$

When you take the time rate of change of the velocities you get accelerations. When going from the velocity equations to the acceleration equations, we now have two time varying terms, ω and θ for each changing angle. This creates a couple of more terms in the acceleration equations. Since accelerations have been shown to lag velocities by 90 degrees, the sin and cos change again.

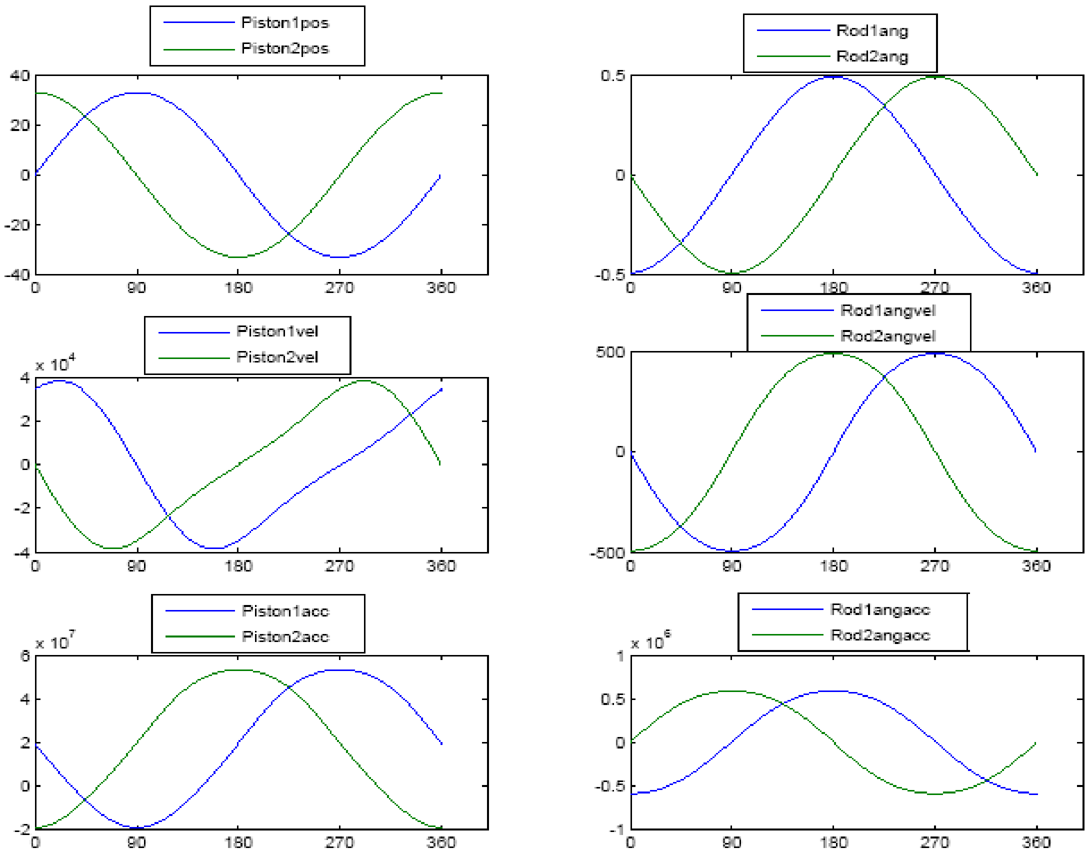
Acceleration (rate of change of velocity):

$\alpha r\cos\theta - \omega^2 r\sin\theta + \alpha_1 c\cos\theta_1 - \omega_1^2 cc\sin\theta_1 = a_1$
$-\alpha r\sin\theta - \omega^2 r\cos\theta – \alpha_1 c\sin\theta_1 – \omega_1^2 c\cos\theta_1 = 0$
$\alpha r\cos\theta - \omega^2 r\sin\theta + \alpha_2 c\cos\theta_2 – \omega_2^2 c\sin\theta_2 = 0$
$-\alpha r\sin\theta - \omega^2 r\cos\theta – \alpha_2 c\sin\theta_2 - \omega_2^2 c\cos\theta_2 = a_2$

Now we have the set of equations that define how the mechanism moves. Rotate the crank, it moves the pistons. Move a piston, it will turn the crank.

I took this set of equations and wrote a computer program in MATLAB to solve them. The programming itself is beyond the scope of this discussion, unless somebody is really interested. It would take some explanation, if you are not familiar with matrix algebra.

I set the crank throw to 33mm (half the stroke length of a VTR engine). I estimated the rod length at 70mm center-center. I started the simulation with the crank angle θ at zero, defining the rod angles and piston positions at this known position, then ran the simulation in 100 steps for one rotation of the crankshaft. The crank was set at a constant rotating speed of 10,000 RPM, which is 1047 radians/second. Here are the resulting positions, velocities and accelerations:




In the above plots, the horizontal axis is the crank angle, in degrees. The vertical axis in the top plots are position form midstroke in mm on the right and radians on the left. The middle plots are mm/sec and radians/sec. The lower plots are mm/sec/sec and radians/sec/sec.

It is interesting to note the asymmetry in the velocity plot for the pistons. This is due to the difference in the relationship between crank angle and rod angle in the top half of the stroke vs. the bottom half. It can also be seen in the acceleration plots. The top hump on the curve is broader and flatter than the bottom half.

It is also interesting that the piston maximum velocity does not occur at midstroke, but slightly later. It took me a while to accept that. But after much deliberation and rederiving the velocity equations, I am fully convinced that it is true. The pistons do not move in simple harmonic motion, because their position is dependant on two different time-varying angles.

Now that we have the motions defined, we can use the masses of the components to compute the forces.